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T-uniqueness of some families of k -chordal matroids

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Abstract

We define k -chordal matroids as a generalization of chordal matroids, and develop tools for proving that some k -chordal matroids are T-unique, that is, determined up to isomorphism by their Tutte polynomials. We apply this theory to wheels, whirls, free spikes, binary spikes, and certain generalizations.

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1. Introduction

The Tutte polynomial [8,10,12] is one of the most studied invariants in matroid theory. While the Tutte polynomial encodes a considerable amount of important information about the matroid, there are many instances of nonisomorphic matroids that have the same Tutte polynomial (see [3] for constructions of large families of nonisomorphic matroids with the same Tutte polynomial). Several recent papers [1,4,16–18] show that certain graphs and matroids are determined by their Tutte polynomials, that is, they are T-unique in the sense of the following definition.

Definition 1. Within the class of graphs without isolated vertices, a graph G is *T-unique* if each graph that has the same Tutte polynomial as G is isomorphic to G . A matroid M is *T-unique* if each matroid that has the same Tutte polynomial as M is isomorphic to M .

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In this paper, we develop several tools that can be of considerable use for showing that particular matroids are T-unique. More specifically, we define k -chordal matroids as a generalization of chordal matroids and we apply the results we develop about k -chordal matroids to prove that certain matroids that arise frequently in structure theory are T-unique.

Section 2 contains relevant background on Tutte polynomials, single-element extensions, and parallel connections. The generalizations of chordal matroids, along with some basic properties of such matroids, are presented in Section 3. In Section 4, we apply the theory developed in Section 3 to show that the following matroids are T-unique: wheels, whirls, and the counterparts obtained by adding the same number of points freely to each nontrivial line of a wheel or whirl. Section 5 defines a generalization of spikes and shows that most of these general spikes are differentiated from all other matroids by their Tutte polynomial; moreover, binary spikes and generalized free spikes are T-unique. Finally, Section 6 gives applications to matroid reconstruction.

We follow the notation and terminology in [20]. In particular, the *girth* $g(M)$ of a matroid M that is not free is the smallest cardinality among circuits of M . We use the term *geometry* for a simple matroid or combinatorial geometry. If needed to avoid ambiguity when several matroids are under discussion, we will use cl_M to denote the closure operator cl of the matroid M .

2. Background

In this section, we review the basic results about Tutte polynomials that are used in this paper, we sketch the parts of the theory of single-element extensions that are needed in Section 4, and we recall the basic facts about parallel connections that are used in Sections 4 and 5.

Recall that the *Tutte polynomial* $t(M; x, y)$ of a matroid M on the ground set S is given by

$$t(M; x, y) = \sum_{A \subseteq S} (x - 1)^{r(M) - r(A)} (y - 1)^{|A| - r(A)}. \quad (1)$$

There are a variety of polynomials that are related to the Tutte polynomial by simple changes of variables; one such polynomial is the rank–cardinality generating function

$$F(M; x, y) = \sum_{A \subseteq S} x^{r(A)} y^{|A|}.$$

Among all such polynomials, the Tutte polynomial has received the most attention, in part because it is the universal matroid invariant that satisfies a deletion–contraction rule (see [7,23]).

We will use the following result about the Tutte polynomial. All statements in this theorem are well known; for the sake of completeness, we prove the two that are less immediate.

Theorem 2. *The following invariants of a matroid M on a set S can be deduced from its Tutte polynomial $t(M; x, y)$:*

- (1) $r(M)$,
- (2) $|S|$,
- (3) for each i with $0 \leq i \leq r(M)$, the number of independent sets of M of cardinality i ,
- (4) the girth $g(M)$,
- (5) the number of circuits of M that have cardinality $g(M)$, and
- (6) for each i with $0 \leq i \leq r(M)$, the largest cardinality among all flats of M of rank i , and the number of rank- i flats of this cardinality.

Whether M is a geometry can be deduced from $t(M; x, y)$. Furthermore, if M is a geometry, one can also deduce the following invariants from $t(M; x, y)$:

- (7) for each integer j with $j \geq 2$, the number of lines (i.e., flats of rank 2) of M that have cardinality j , and
- (8) the number of 4-circuits of M .

Proof. Assertions (1)–(6) are well known and easy to see. To address assertions (7) and (8), assume that M is a geometry. From assertion (6), we can deduce the number, say k , of lines of the largest cardinality, say t . For i with $2 \leq i \leq t$, subtract $k \binom{t}{i}$ from the coefficient of $x^2 y^i$ in $F(M; x, y)$. From the resulting polynomial, one can deduce the second largest cardinality among lines of M , and the number of lines of M having this cardinality. Applying this idea recursively gives assertion (7). For assertion (8), note that the number of sets of size four and rank three is the coefficient of $x^3 y^4$ in $F(M; x, y)$, and such sets are of two types: 4-circuits and 4-sets that contain a unique 3-circuit. By assertion (7), we know the number of 4-sets that contain a unique 3-circuit, so assertion (8) follows. \square

The idea used to prove assertion (7) of Theorem 2 can be extended to yield the following generalization of that assertion; this extension is used in Section 5. (See [8, Proposition 5.9] and the discussion beginning on p. 195 of that paper for a stronger formulation of Theorem 3.)

Theorem 3. *For a rank- n matroid M and any integer i with $0 \leq i \leq n$, let c_i be the largest cardinality among rank- i flats of M . Then for each i with $1 \leq i \leq n$ and each j with $c_{i-1} < j \leq c_i$, the number of flats of M having rank i and cardinality j can be deduced from the Tutte polynomial.*

Crapo's theory of single-element extensions [11,13,20] plays a role in Section 4; we briefly review the relevant ideas and terminology here. Assume that the matroid M^+ on the ground set $S \cup e$ is a *single-element extension* of the matroid M on the ground set S , i.e., M is the restriction of M^+ to S . The flats of M^+ are of the form A or $A \cup e$ where A ranges over the flats of M . In particular, the flats of M are partitioned into the following three collections that completely determine M^+ :

$$\mathcal{M} = \{A \mid A \cup e \text{ is a flat of } M^+ \text{ but } A \text{ is not a flat of } M^+\},$$

$$\mathcal{C} = \{A \mid A \text{ is a flat of } M^+ \text{ but } A \cup e \text{ is not a flat of } M^+\},$$

$$\mathcal{I} = \{A \mid \text{both } A \text{ and } A \cup e \text{ are flats of } M^+\}.$$

The collection \mathcal{M} is a filter, that is, if X is in \mathcal{M} and Y is a flat of M with $X \subseteq Y$, then Y is in \mathcal{M} . Furthermore, \mathcal{M} has this property: if A and B are in \mathcal{M} with A and B covering the flat $A \cap B$, then $A \cap B$ is in \mathcal{M} . Any filter of flats with this property is called a *modular filter*. The collection \mathcal{C} is called the *collar* of the extension. A flat A is in the collar if and only if A is not in the modular filter but is covered by a flat in the modular filter. The collection \mathcal{I} is the ideal of flats in neither the modular filter nor the collar. Thus, from \mathcal{M} we can find both \mathcal{C} and \mathcal{I} .

A fundamental result about single-element extensions is the following. Not only does every single-element extension give rise to a modular filter, but the converse holds: any modular filter \mathcal{M} of M gives rise to a single-element extension of M . To get the flats of the single-element extension corresponding to \mathcal{M} , we find \mathcal{C} and \mathcal{I} from \mathcal{M} as above and construct the flats as specified by these three collections.

Our concern is with principal extensions. It is easy to see that for any flat X of M , the set

$$\mathcal{M}_X = \{A \mid A \text{ is a flat of } M \text{ with } X \subseteq A\}$$

is a modular filter of M . The corresponding extension of M is denoted by $M +_X e$ and is called the *principal extension of M with respect to X* . The extension $M +_X e$ makes precise the notion of adding the point e freely to the flat X of M . Lemma 4 follows immediately from our discussion.

Lemma 4. *For any flat X of M and any subset A of S , the element e is in the closure of A in $M +_X e$ if and only if the closure of A in M contains X .*

We conclude this section with the basic results about parallel connections of matroids [6, 20] that are used in Sections 4 and 5. All matroids of interest in this paper are geometries; the discussion below reflects the minor streamlining of the theory of parallel connections that results from not having loops.

Assume that M and N are loopless matroids with ground sets S and T , respectively, and that $S \cap T = \{p\}$. The *parallel connection of M and N with respect to the basepoint p* is the matroid $P(M, N)$ whose collection of flats is

$$\{X \mid X \subseteq S \cup T, X \cap S \text{ is a flat of } M, \text{ and } X \cap T \text{ is a flat of } N\}.$$

The collection of circuits of the parallel connection $P(M, N)$ is

$$\mathcal{C}_M \cup \mathcal{C}_N \cup \{(C \cup C') - p \mid C \in \mathcal{C}_M, C' \in \mathcal{C}_N, \text{ and } p \in C \cap C'\},$$

where \mathcal{C}_M and \mathcal{C}_N are the collections of circuits of M and N , respectively.

Theorem 5 is part (ii) of [20, Theorem 7.1.16].

Theorem 5. Assume that M is a connected matroid and that p is in the ground set of M . If $M/p = M_1 \oplus M_2$, where M_1 and M_2 have ground sets S_1 and S_2 , respectively, then $M = P(M|(S_1 \cup p), M|(S_2 \cup p))$.

We will use the following corollary of Theorem 5.

Corollary 6. Assume that M is a rank- n geometry and that the ground set of M is the union of $n - 1$ lines, $\ell_1, \ell_2, \dots, \ell_{n-1}$, where

$$\ell_1 \cap \ell_2 = \{p_1\}, \ell_2 \cap \ell_3 = \{p_2\}, \dots, \ell_{n-2} \cap \ell_{n-1} = \{p_{n-2}\}$$

and the $n - 2$ points p_1, p_2, \dots, p_{n-2} are distinct. Then M is formed by taking the parallel connection of ℓ_1 and ℓ_2 with respect to p_1 , and then the parallel connection of the resulting matroid and ℓ_3 with respect to p_2 , and so on.

Proof. A rank calculation shows that $(\ell_1 \cup \ell_2 \cup \dots \cup \ell_{n-2}) - p_{n-2}$ and $\ell_{n-1} - p_{n-2}$ are complementary separators of M/p_{n-2} . The result now follows from Theorem 5 by induction on n . \square

3. $[k_1, k_2]$ -chordal matroids

In [2,26], a binary matroid M is said to be chordal if each circuit C of M that has four or more elements has a chord, that is, an element $e \notin C$ so that for some partition of C into two blocks C_1 and C_2 with $|C_1|, |C_2| \geq 2$, both $C_1 \cup e$ and $C_2 \cup e$ are circuits of M . Chordal matroids are the natural generalization of chordal graphs, which give rise to the graphic matroids that are supersolvable [24]; chordal graphs are also the topic of much research in graph theory. We are interested in the following natural variations on the notion of a chordal matroid.

Definition 7. A circuit C of a matroid M is *chordal* if there are circuits C_1 and C_2 of M and an element x in $C_1 \cap C_2$ such that $|C_1|, |C_2| < |C|$ and $C = (C_1 \cup C_2) - x$.

Let $[k_1, k_2]$ be an interval of integers with $k_1 > g(M)$ and $k_2 \leq r(M) + 1$. A matroid M is $[k_1, k_2]$ -chordal if each circuit C of M with $k_1 \leq |C| \leq k_2$ is chordal.

A matroid M is k -chordal if it is $[g(M) + 1, k]$ -chordal.

Note that only circuits with four or more elements can be chordal. Thus, only geometries can be k -chordal. We will be most interested in geometries M that are $r(M)$ -chordal.

For a binary geometry M of girth 3, the notions of chordal (as defined in [2,26]) and $(r(M) + 1)$ -chordal are the same. Thus, the cycle matroid of the complete graph K_n is n -chordal. Note that the cycle matroid of the complete bipartite graph $K_{m,n}$ is $(m + n)$ -chordal but not chordal. An easy argument based on linear combinations shows that any projective geometry M is $(r(M) + 1)$ -chordal; likewise, any affine geometry M over a field

of characteristic not 2 is $(r(M) + 1)$ -chordal. Also note that the truncation of a $[k_1, k_2]$ -chordal matroid to rank h is $[k_1, \min(k_2, h + 1)]$ -chordal.

Our interest in $r(M)$ -chordal matroids M arises from the following theorem.

Theorem 8. Assume M and M' are matroids on the ground sets S and S' , respectively, and M is $r(M)$ -chordal. Assume $\phi: S \rightarrow S'$ is a bijection such that for every circuit C of M with $|C| = g(M)$, its image $\phi(C)$ is a circuit of M' . If either of the following two conditions holds, then M and M' are isomorphic and ϕ is an isomorphism.

- (a) The matroids M and M' have the same girth, the same number of circuits of size $g(M)$, and, for each integer i with $g(M) + 1 \leq i \leq r(M)$, the same number of independent sets of cardinality i .
- (b) $t(M; x, y) = t(M'; x, y)$.

Proof. By Theorem 2, condition (b) implies condition (a), so we focus on (a). We show that a subset C of S is a circuit of M if and only if $\phi(C)$ is a circuit of M' . Since the spanning circuits are precisely the sets of size $r(M) + 1$ that do not contain smaller circuits, it suffices to prove this statement in the case that $|C| \leq r(M)$. We induct on $|C|$.

The base case is $|C| = g(M)$. By hypothesis, for every circuit C of M with $|C| = g(M)$, its image $\phi(C)$ is a circuit of M' . The converse follows since M and M' have the same number of circuits of size $g(M)$.

Assume C is a circuit of M with $|C| = i$ and $g(M) < i \leq r(M)$. Since M is $r(M)$ -chordal, there are circuits C_1 and C_2 of M and an element x in $C_1 \cap C_2$ such that $|C_1|, |C_2| < |C|$ and $C = (C_1 \cup C_2) - x$. Now $\phi(C) = (\phi(C_1) \cup \phi(C_2)) - \phi(x)$. By the inductive assumption, both $\phi(C_1)$ and $\phi(C_2)$ are circuits of M' , so by circuit elimination, $\phi(C)$ contains a circuit, say C' , of M' . If C' were properly contained in $\phi(C)$, then the inductive assumption would give proper containment of the circuit $\phi^{-1}(C')$ in the circuit C , which is impossible. Thus, $\phi(C)$ is a circuit of M' .

For the converse, it suffices to show that M and M' have the same number of i -circuits. Since ϕ is a bijection, $|S| = |S'|$, so S and S' have the same number of i -subsets. By assumption, M and M' have the same number of independent sets of cardinality i , and therefore the same number of dependent sets of cardinality i . Thus, it suffices to show that M and M' have the same number of i -subsets that properly contain a circuit; however, by the inductive assumption the mapping ϕ provides a bijection between such sets, thereby completing the proof. \square

We will also use the following result about k -chordal matroids.

Theorem 9. Assume M and M' are matroids on the ground sets S and S' , respectively, and M is k -chordal. Assume $\phi: S \rightarrow S'$ is a bijection such that for every circuit C of M with $|C| = g(M)$, its image $\phi(C)$ is a circuit of M' . If either of the following two conditions holds, then M' is k -chordal.

- (a) *The matroids M and M' have the same girth, the same number of circuits of size $g(M)$, and, for each integer i with $g(M) + 1 \leq i \leq k + 1$, the same number of independent sets of cardinality i .*
- (b) $t(M; x, y) = t(M'; x, y)$.

Furthermore, ϕ is an isomorphism of the truncations of M and M' to rank k . The map ϕ is a bijection between the chordal $(k + 1)$ -circuits of M and the chordal $(k + 1)$ -circuits of M' . Also, M and M' have the same number of nonchordal circuits of size $k + 1$.

Proof. Again we focus on condition (a). Note that the induction argument used in the proof of Theorem 8 works for i with $g(M) \leq i \leq k$; this means that the map ϕ is a bijection from the set of circuits of M of size at most k onto the set of circuits in M' of size at most k . From this we deduce that M' is k -chordal. The assertion about truncations follows from Theorem 8.

The argument in the proof of Theorem 8 also shows that for each chordal circuit C of M with $|C| = k + 1$, its image $\phi(C)$ is a chordal circuit of M' . Note that ϕ^{-1} satisfies the hypotheses of the theorem. From this observation, it follows that if $\phi(C)$ is a chordal circuit of M' of size $k + 1$, then C is a chordal circuit of M .

As in the proof of Theorem 8, ϕ gives a bijection between the sets of size $k + 1$ that properly contain circuits. Also, M and M' have the same number of dependent sets of size $k + 1$, and thus the same number of $(k + 1)$ -circuits. This, together with the conclusion of the last paragraph, shows that M and M' have the same number of nonchordal circuits of size $k + 1$. \square

We end this section with observations on parallel connections of matroids. The proofs are straightforward and hence omitted.

Theorem 10. *Assume the geometries M and N are $(r(M) + 1)$ -chordal and $(r(N) + 1)$ -chordal, respectively, and that the ground sets of M and N intersect in one element. Then $P(M, N)$ is $[\max(g(M), g(N)) + 1, r(M) + r(N)]$ -chordal. If M and N have the same girth, then $P(M, N)$ is $(r(M) + r(N))$ -chordal.*

4. Wheels, whirls, and generalizations

The *rank- n wheel* \mathcal{W}_n is the graph that consists of an n -cycle, the *rim*, and one additional vertex, the *hub*, that is adjacent to each vertex on the rim. Label the edges that are incident with the hub as b_0, b_1, \dots, b_{n-1} ; these edges are the *spokes*. Label the rim edges as a_0, a_1, \dots, a_{n-1} so that for each i , the edges b_i, a_i, b_{i+1} form a 3-circuit; here and below, subscripts are interpreted modulo n . The rim edges form a circuit-hyperplane of the cycle matroid $M(\mathcal{W}_n)$ of \mathcal{W}_n . The matroid obtained by relaxing this circuit-hyperplane is the *rank- n whirl*, denoted \mathcal{W}^n .

Wheels and whirls play a major role in matroid structure theory (see [20,21]). In this section, we define generalizations of wheels and whirls and show that these matroids are T-unique.

It is easy to see that the circuits of the n -wheel \mathcal{W}_n are of two types:

- (i) $\{a_0, a_1, \dots, a_{n-1}\}$, and
- (ii) $\{b_i, a_i, a_{i+1}, \dots, a_{j-1}, b_j\}$ for any distinct integers i and j in $\{0, 1, \dots, n-1\}$.

It follows that the circuits of the n -whirl \mathcal{W}^n are also of two types:

- (i') $\{b_i, a_0, a_1, \dots, a_{n-1}\}$ for any i in $\{0, 1, \dots, n-1\}$, and
- (ii') $\{b_i, a_i, a_{i+1}, \dots, a_{j-1}, b_j\}$ for any distinct integers i and j in $\{0, 1, \dots, n-1\}$.

We now define the (n, t) -wheel, $\mathcal{W}_{n,t}$, and the (n, t) -whirl, $\mathcal{W}^{n,t}$, for any integer $t \geq 3$; the matroids $\mathcal{W}_{n,3}$ and $\mathcal{W}^{n,3}$ are $M(\mathcal{W}_n)$ and \mathcal{W}^n , respectively. The (n, t) -wheel $\mathcal{W}_{n,t}$ is obtained from $M(\mathcal{W}_n)$ by adding the following points: for each i with $0 \leq i \leq n-1$, the $t-3$ points $x_{i,1}, x_{i,2}, \dots, x_{i,t-3}$ are added freely to the line $\text{cl}(\{b_i, b_{i+1}\})$. More precisely, extend $M(\mathcal{W}_n)$ by adding $x_{0,1}$, using the principal modular filter of $M(\mathcal{W}_n)$ generated by $\text{cl}_{M(\mathcal{W}_n)}(\{b_0, b_1\})$; let M_1 denote the resulting matroid; then extend M_1 by adding $x_{1,1}$, using the principal modular filter of M_1 generated by $\text{cl}_{M_1}(\{b_1, b_2\})$, and so on, doing this for each line $\text{cl}(\{b_i, b_{i+1}\})$ of $M(\mathcal{W}_n)$, adding each of the points $x_{0,1}, x_{1,1}, \dots, x_{n-1,1}$ in turn; the resulting matroid is $\mathcal{W}_{n,4}$. In general, we obtain $\mathcal{W}_{n,t}$ by extending $\mathcal{W}_{n,t-1}$, first adding $x_{0,t-3}$, using the principal modular filter of $\mathcal{W}_{n,t-1}$ generated by $\text{cl}_{\mathcal{W}_{n,t-1}}(\{b_0, b_1\})$, and then adding the points $x_{1,t-3}, x_{2,t-3}, \dots, x_{n-1,t-3}$ in a similar way. Although we have described these extensions of $M(\mathcal{W}_n)$ with a particular ordering of the points $x_{i,j}$, for $0 \leq i \leq n-1$ and $1 \leq j \leq t-3$, that the proof of Lemma 12 does not rely on this order implies that this matroid is independent of the order in which these extensions are carried out. The (n, t) -whirl $\mathcal{W}^{n,t}$ is defined in a similar manner; specifically, the (n, t) -whirl $\mathcal{W}^{n,t}$ is obtained from $M(\mathcal{W}^n)$ by adding $n(t-3)$ points, with $x_{i,1}, x_{i,2}, \dots, x_{i,t-3}$ added freely to the line $\text{cl}(\{b_i, b_{i+1}\})$. Note that both $\mathcal{W}_{n,t}$ and $\mathcal{W}^{n,t}$ have rank n and $(t-1)n$ points.

The $(n, 4)$ -whirl $\mathcal{W}^{n,4}$ is also known as the swirl. Swirls were first defined in [22] and they play a key role in [3].

To apply Theorems 8 and 9 to $\mathcal{W}_{n,t}$ and $\mathcal{W}^{n,t}$, we need the following theorem.

Theorem 11. *The (n, t) -wheel $\mathcal{W}_{n,t}$ is $(n-1)$ -chordal. The (n, t) -whirl $\mathcal{W}^{n,t}$ is $(n+1)$ -chordal.*

Proof. The result follows immediately from the analysis of the circuits of $\mathcal{W}_{n,t}$ and $\mathcal{W}^{n,t}$ in Lemma 12. \square

Lemma 12. *The circuits of the (n, t) -wheel $\mathcal{W}_{n,t}$ are the sets C that satisfy one of the following four properties.*

- (I) *The set C is a 3-subset of the line $\text{cl}(\{b_j, b_{j+1}\})$ for some j .*
- (II) *For some s and k with $0 \leq s \leq n-1$ and $1 < k < n$, the set C consists precisely of the following points:*
 - (a) *any two points from $\text{cl}(\{b_s, b_{s+1}\}) - b_{s+1}$,*
 - (b) *any two points from $\text{cl}(\{b_{s+k-1}, b_{s+k}\}) - b_{s+k-1}$, and*

- (c) for each j with $1 \leq j \leq k-2$, any single point from $\text{cl}(\{b_{s+j}, b_{s+j+1}\}) - \{b_{s+j}, b_{s+j+1}\}$.
- (III) $C = \{a_0, a_1, \dots, a_{n-1}\}$.
- (IV) (a) $|C| = n+1$,
 (b) for all i , we have $|C \cap \{a_i, x_{i,1}, x_{i,2}, \dots, x_{i,t-3}\}| \geq 1$, and
 (c) $\{a_0, a_1, \dots, a_{n-1}\} \not\subseteq C$.

The circuits of the (n, t) -whirl $\mathcal{W}^{n,t}$ are the sets C that satisfy (I), (II), or

- (III') (a) $|C| = n+1$, and
 (b) for all i , we have $|C \cap \{a_i, x_{i,1}, x_{i,2}, \dots, x_{i,t-3}\}| \geq 1$.

In particular, no circuit of the (n, t) -whirl $\mathcal{W}^{n,t}$ contains exactly one point from each line $\text{cl}(\{b_i, b_{i+1}\})$.

Proof. By Corollary 6, the deletion $\mathcal{W}_{n,t} \setminus \{a_i, x_{i,1}, x_{i,2}, \dots, x_{i,t-3}\}$ is a parallel connection of $n-1$ lines, $\text{cl}(\{b_{i+1}, b_{i+2}\}), \text{cl}(\{b_{i+2}, b_{i+3}\}), \dots, \text{cl}(\{b_{i-1}, b_i\})$, with respect to the basepoints $b_{i+2}, b_{i+3}, \dots, b_{i-1}$. From the structure of the circuits in parallel connections of lines, it follows that the circuits of $\mathcal{W}_{n,t}$ that, for some i , do not contain any of the points $a_i, x_{i,1}, x_{i,2}, \dots, x_{i,t-3}$ are the sets C that satisfy either property (I) or (II). All other circuits contain at least one point from each set $\{a_i, x_{i,1}, x_{i,2}, \dots, x_{i,t-3}\}$. Since $\mathcal{W}_{n,t}$ is an extension of $M(\mathcal{W}_n)$, the circuits of $\mathcal{W}_{n,t}$ that do not contain any of the new points $x_{i,j}$ are the same as those of $M(\mathcal{W}_n)$. Therefore the set C in (III) is a circuit of $\mathcal{W}_{n,t}$. We claim that (III) gives the only n -circuit that contains one element from each of the sets $\{a_i, x_{i,1}, x_{i,2}, \dots, x_{i,t-3}\}$. Indeed, assume that C is an n -circuit that contains one element from each of these sets, and that $x_{i,j}$ is the last element in C that is added in the sequence of single-element extensions that yield $\mathcal{W}_{n,t}$. Both $(C - x_{i,j}) \cup b_i$ and $(C - x_{i,j}) \cup b_{i+1}$ are bases of the parallel connection $\mathcal{W}_{n,t} \setminus \{a_i, x_{i,1}, x_{i,2}, \dots, x_{i,t-3}\}$ of $n-1$ lines, so the closure of $C - x_{i,j}$ is not in the principal modular filter that was used when $x_{i,j}$ was added, and so, by Lemma 4, the point $x_{i,j}$ is not in the closure of $C - x_{i,j}$ in $\mathcal{W}_{n,t}$. It follows that the remaining circuits are the sets C that satisfy property (IV). The structure of the circuits of the (n, t) -whirl $\mathcal{W}^{n,t}$ can be deduced in a similar way. \square

Note that all circuits that satisfy properties (IV) and (III') and some of the circuits that satisfy property (II) are spanning circuits. Also, a circuit that satisfies property (IV) or (III') contains at most one point b_i among b_0, b_1, \dots, b_{n-1} .

We now turn to the main result of this section. We give a characterization of (n, t) -wheels and (n, t) -whirls by numerical invariants; this characterization implies that (n, t) -wheels and (n, t) -whirls are T-unique. Note that wheels \mathcal{W}_n were already known to be T-unique as graphs [17]. Also note that $\mathcal{W}_{3,3}$ and $\mathcal{W}^{3,3}$ have particularly simple characterizations: it is easy to check that $\mathcal{W}_{3,3}$ and $\mathcal{W}^{3,3}$ are the only rank-3 geometries on six points for which the number of 3-point lines is, respectively, four and three. We omit these cases in Theorem 13 since condition (3) of Theorem 13 does not hold in the case of $\mathcal{W}_{3,3}$.

Theorem 13. Assume that n and t are integers with $n, t \geq 3$ and either $n > 3$ or $t > 3$. Assume that M is a geometry on the ground set S that satisfies the following properties:

- (1) $r(M) = n$,
- (2) $|S| = (t - 1)n$,
- (3) there are exactly n lines $\ell_1, \ell_2, \dots, \ell_n$ with $|\ell_i| = t$,
- (4) for s with $2 \leq s \leq n - 1$, flats of rank s have at most $(s - 1)(t - 1) + 1$ points, and
- (5) for each s with $3 \leq s \leq n$, the geometry M has the same number of independent sets of size s as $\mathcal{W}_{n,t}$.

Then M is isomorphic to the (n, t) -wheel $\mathcal{W}_{n,t}$.

Assume that M is a geometry on the ground set S that satisfies properties (1)–(4) and

- (5') for each s with $3 \leq s \leq n$, the geometry M has the same number of independent sets of size s as $\mathcal{W}^{n,t}$.

Then M is isomorphic to the (n, t) -whirl $\mathcal{W}^{n,t}$. In particular, (n, t) -wheels and (n, t) -whirls are T -unique.

Proof. We first show that M is a ring of t -point lines in the following sense: M has a basis p_0, p_1, \dots, p_{n-1} such that each of the lines $\text{cl}(\{p_i, p_{i+1}\})$, for i with $0 \leq i \leq n - 1$, has t points and these lines contain all points of M . Towards this end, we introduce several more definitions. A sequence $\ell'_1, \ell'_2, \dots, \ell'_k$ of t -point lines *intersects well* if for each i with $1 < i \leq k$, there is a j such that $j < i$ and $\ell'_j \cap \ell'_i \neq \emptyset$. An *ordered component* of M is a maximal sequence of t -point lines that intersects well. Our interest in ordered components is more in the collections of lines rather than in the ordering. Note that each maximal component with at least two lines has more than one ordering with respect to which it intersects well. We say that M has a *unique ordered component* if all ordered components of M use all t -point lines of M .

To show that M is a ring of t -point lines, we prove several properties about sequences that intersect well.

- (13.1) Assume $1 \leq k \leq n - 2$. Every sequence $\ell'_1, \ell'_2, \dots, \ell'_k$ of t -point lines that intersects well has rank $k + 1$ and contains $(t - 1)k + 1$ points. Furthermore, $\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_k$ is a flat and $\ell'_1, \ell'_2, \dots, \ell'_k$ are the only nontrivial lines of $M|(\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_k)$.

Proof. We prove the following four statements by induction on k :

- (R_k) $r(\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_k) = k + 1$,
- (S_k) $|\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_k| = (t - 1)k + 1$,
- (C_k) $\text{cl}(\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_k) = \ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_k$,
- (L_k) $\ell'_1, \ell'_2, \dots, \ell'_k$ are the only nontrivial lines of $M|(\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_k)$.

The case $k = 1$ is trivial. Assume that $\ell'_1, \ell'_2, \dots, \ell'_{k-1}, \ell'_k$ is a sequence of t -point lines that intersects well. Note that $\ell'_1, \ell'_2, \dots, \ell'_{k-1}$ is also a sequence of t -point lines that

intersects well. Statements (C_{k-1}) , (L_{k-1}) , and the definition of intersecting well imply that ℓ'_k intersects $\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_{k-1}$ in exactly one point. Statements (R_k) and (S_k) follow now immediately from (R_{k-1}) , (C_{k-1}) , and (L_{k-1}) . From (R_k) , (S_k) , and assumption (4), we get (C_k) . Since ℓ'_k intersects $\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_{k-1}$ in exactly one point, it follows from this and statements (L_{k-1}) and (C_k) that the only nontrivial line of $M|(\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_k)$ other than $\ell'_1, \ell'_2, \dots, \ell'_{k-1}$ is ℓ'_k , as asserted in (L_k) . \square

(13.2) A sequence $\ell'_1, \ell'_2, \dots, \ell'_{n-1}$ of t -point lines that intersects well has rank n and the remaining t -point line ℓ'_n must intersect ℓ'_{n-1} in a point that is not in $\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_{n-2}$. Furthermore, ℓ'_n must intersect one line from among $\ell'_1, \ell'_2, \dots, \ell'_{n-2}$. Thus, $\ell'_1, \ell'_2, \dots, \ell'_{n-1}, \ell'_n$ intersects well.

Proof. The rank assertion follows as in the proof of (13.1). By comparing the size of S with that of $\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_{n-1}$ and ℓ'_n , it follows that ℓ'_n intersects $\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_{n-1}$ in at least two points. Since the distinct lines ℓ'_n and ℓ'_{n-1} can intersect in at most one point and since $\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_{n-2}$ is a flat of M whose only nontrivial lines are $\ell'_1, \ell'_2, \dots, \ell'_{n-2}$, it follows that ℓ'_n intersects $\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_{n-1}$ in exactly two points and that ℓ'_n intersects ℓ'_{n-1} in one point. \square

(13.3) The geometry M has a unique ordered component.

Proof. Assume M has h ordered components; let c_j be the number of t -point lines in the j th ordered component. Thus, $\sum_{j=1}^h c_j = n$. If $h > 1$, then by (13.2) each c_j is less than $n - 1$, so conclusion (13.1) applies to each ordered component. Thus, the number of points in M is at least

$$\sum_{j=1}^h ((t-1)c_j + 1) = (t-1) \left(\sum_{j=1}^h c_j \right) + h = (t-1)n + h,$$

which exceeds $(t-1)n$. This contradiction shows that h is 1, as claimed. \square

(13.4) The geometry M is a ring of t -point lines.

Proof. From (13.3), we know that M has a unique ordered component; assume that the sequence $\ell_1, \ell_2, \dots, \ell_n$ of t -point lines intersects well. We first claim that we may assume that the lines $\ell_1, \ell_2, \dots, \ell_n$ are ordered so that for i with $1 \leq i < n$, each intersection $\ell_i \cap \ell_{i+1}$ is nonempty and these points of intersection are distinct. We show this by induction. Since this sequence intersects well, it follows that the intersection $\ell_1 \cap \ell_2$ is a point p_1 . Assume that $j < n - 1$ and that the sequence $\ell_1, \ell_2, \dots, \ell_j$ has the property that for each i with $i < j$, the intersection $\ell_i \cap \ell_{i+1}$ consists of one point p_i , and p_1, p_2, \dots, p_{j-1} are distinct. We claim that there is a line ℓ_k with $k > j$ with the intersection $\ell_j \cap \ell_k$ being a point other than p_{j-1} . Since ℓ_j intersects $\ell_1 \cup \ell_2 \cup \dots \cup \ell_{j-1}$ in exactly one point, if there were no such ℓ_k , then the sequence $\ell_1, \ell_2, \dots, \ell_{j-1}, \ell_{j+1}, \dots, \ell_n, \ell_j$ would intersect well and ℓ_j would contain

only one point from $\bigcup_{i \neq j} \ell_i$ contrary to (13.2). Thus we may assume that $\ell_j \cap \ell_{j+1}$ is a point p_j different from p_{j-1} . By (13.2), ℓ_n intersects ℓ_{n-1} in a point p_{n-1} with $p_{n-1} \neq p_{n-2}$.

Note that the sequence $\ell_{n-1}, \ell_{n-2}, \dots, \ell_1, \ell_n$ intersects well. It therefore follows from (13.2) that ℓ_n intersects ℓ_1 in a point p_0 with $p_0 \neq p_1$.

Since p_0, p_1, \dots, p_{n-1} span the lines $\ell_1, \ell_2, \dots, \ell_n$ and, by assumption (2), these lines contain all points of M , it follows that $\{p_0, p_1, \dots, p_{n-1}\}$ is a basis and M is a ring of t -point lines. \square

Let $\{p_0, p_1, \dots, p_{n-1}\}$ be a basis that shows that M is a ring of t -point lines. From Corollary 6, it follows that $M \setminus (\text{cl}(\{p_i, p_{i+1}\}) - \{p_i, p_{i+1}\})$ is a parallel connection of t -point lines. Thus for $n > 3$, the 3-circuits of M are precisely the 3-element subsets of the lines $\text{cl}(\{p_j, p_{j+1}\})$.

Now assume condition (5') holds. Since M is a geometry, in the case $n = 3$, this condition implies that M has the same number of 3-circuits as $\mathcal{W}^{3,t}$. This conclusion, along with the fact that all 3-element subsets of the lines $\text{cl}(\{p_i, p_{i+1}\})$ are 3-circuits, implies that the 3-circuits of M are precisely the 3-element subsets of the lines $\text{cl}(\{p_i, p_{i+1}\})$. Therefore, we can strengthen the conclusion of the last paragraph: for any $n \geq 3$, the 3-circuits of M are precisely the 3-element subsets of the lines $\text{cl}(\{p_j, p_{j+1}\})$. Thus, M and the (n, t) -whirl $\mathcal{W}^{n,t}$ have the same number of 3-circuits, and any bijection of the ground set of $\mathcal{W}^{n,t}$ with the ground set S of M that maps b_i in $\mathcal{W}^{n,t}$ to p_i in M and that maps the line $\text{cl}(\{b_i, b_{i+1}\})$ of $\mathcal{W}^{n,t}$ to the line $\text{cl}_M(\{p_i, p_{i+1}\})$ of M gives a bijection between the 3-circuits of $\mathcal{W}^{n,t}$ and the 3-circuits of M . This observation, Theorem 11, and Theorem 8 complete the proof in the case of the (n, t) -whirl $\mathcal{W}^{n,t}$.

Now assume condition (5) holds and assume $n > 3$. The same argument as above shows that any bijection of the ground set of the (n, t) -wheel $\mathcal{W}_{n,t}$ with the ground set S of M that maps b_i in $\mathcal{W}_{n,t}$ to p_i in M and that maps the line $\text{cl}(\{b_i, b_{i+1}\})$ of $\mathcal{W}_{n,t}$ to the line $\text{cl}_M(\{p_i, p_{i+1}\})$ of M gives a bijection between the 3-circuits of $\mathcal{W}_{n,t}$ and the 3-circuits of M . By Theorems 9 and 11, it follows that M , like $\mathcal{W}_{n,t}$, has precisely one nonchordal n -circuit. That $M \setminus (\text{cl}(\{p_i, p_{i+1}\}) - \{p_i, p_{i+1}\})$ is a parallel connection of t -point lines allows us to conclude that this nonchordal n -circuit of M must contain precisely one point in each set $\text{cl}(\{p_i, p_{i+1}\}) - \{p_i, p_{i+1}\}$. Since the bijection ϕ of Theorem 9 can be chosen to map the circuit a_0, a_1, \dots, a_{n-1} of $\mathcal{W}_{n,t}$ to this nonchordal n -circuit of M , the map ϕ gives a bijection between all nonspanning circuits of $\mathcal{W}_{n,t}$ and those of M , which suffices to complete the proof in the case of the (n, t) -wheel $\mathcal{W}_{n,t}$ for $n > 3$.

The case of $n = 3$ follows using similar ideas. In particular, from condition (5), it follows that there is exactly one 3-circuit in addition to those arising from the three t -point lines. From this, it is easy to construct a bijection of the ground set of $\mathcal{W}_{3,t}$ with the ground set of M that gives a bijection between the nonspanning circuits.

Finally, note that, by Theorem 2, conditions (1)–(5) and (5') can be deduced from the Tutte polynomial. \square

5. Spikes and generalizations

In this section, we generalize the notion of a spike as defined in [21] and we prove a number of properties about these matroids. In particular, we show that a large class of these generalized spikes are distinguished from all other matroids by their Tutte polynomials and we show that binary spikes and the generalizations of free spikes are T-unique. We start by defining this more general notion of a spike.

Definition 14. Assume n, s , and t are integers with $n \geq 3$, $s \geq n - 1$, and $t \geq 3$. An (n, s, t) -spike with tip a is a rank- n geometry whose ground set is the union of s lines $\ell_1, \ell_2, \dots, \ell_s$ for which the following properties hold:

- (1) $a \in \ell_i$ for $1 \leq i \leq s$,
- (2) $|\ell_i| = t$ for $1 \leq i \leq s$, and
- (3) for any $k < n$ and any integers $1 \leq i_1 < i_2 < \dots < i_k \leq s$, we have $r(\ell_{i_1} \cup \ell_{i_2} \cup \dots \cup \ell_{i_k}) = k + 1$.

Consistent with [21], an $(n, n, 3)$ -spike will be called an n -spike. The following $n \times (2n + 1)$ matrix D over the field $\text{GF}(2)$ represents an n -spike; the last column corresponds to the tip:

$$D = \left(\begin{array}{ccccc|ccccc|c} 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 & 0 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & \dots & 0 & 1 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 & 1 & 1 & \dots & 0 & 1 \end{array} \right). \quad (2)$$

Throughout this section, if M is an (n, s, t) -spike, we let $\ell_1, \ell_2, \dots, \ell_s$ denote the lines through the tip a of M . Let S be the ground set of an (n, s, t) -spike M and let the set \mathcal{Z}_n be defined as follows:

$$\mathcal{Z}_n = \{X \mid X \subseteq S - a, |X| = n, \text{ and } |X \cap \ell_i| \leq 1 \text{ for } 1 \leq i \leq s\}. \quad (3)$$

Thus, \mathcal{Z}_n is the collection of all n -sets X such that for some i_1, i_2, \dots, i_n with $1 \leq i_1 < i_2 < \dots < i_n \leq s$, the set X consists of precisely one point in each of the sets $\ell_{i_1} - a, \ell_{i_2} - a, \dots, \ell_{i_n} - a$.

It follows from condition (3) of Definition 14 that each set X in \mathcal{Z}_n is either a basis or a circuit of M . If $s = n$, the circuits in \mathcal{Z}_n are necessarily circuit-hyperplanes of M , but if $s > n$, the circuits in \mathcal{Z}_n might not be circuit-hyperplanes.

Condition (3) of Definition 14 also implies that there are only three types of nonspanning circuits of an (n, s, t) -spike, namely, the circuits in \mathcal{Z}_n , all 3-subsets of the lines ℓ_i , and, if $n > 3$, all sets of the form $A \cup B$ where A and B are 2-subsets of any two distinct sets $\ell_i - a$ and $\ell_j - a$, respectively. The *free* (n, s, t) -spike is the (n, s, t) -spike in which there are no circuits in \mathcal{Z}_n ; for each triple n, s, t , there is precisely one free (n, s, t) -spike. From our observations on nonspanning circuits, it follows that the restriction

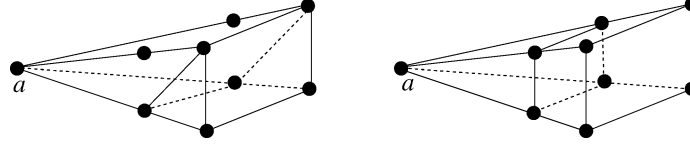


Fig. 1. Nonisomorphic 4-spikes with the same Tutte polynomial.

$M|(\ell'_1 \cup \ell'_2 \cup \dots \cup \ell'_{n-1})$ of an (n, s, t) -spike to any $n - 1$ of the lines through the tip is the parallel connection of $\ell'_1, \ell'_2, \dots, \ell'_{n-1}$ with respect to the common basepoint a . In particular, an $(n, n - 1, t)$ -spike is precisely such a parallel connection. Thus, (n, s, t) -spikes are $(n - 1)$ -chordal, the free (n, s, t) -spike is n -chordal, and the $(n, n - 1, t)$ -spike is $(n + 1)$ -chordal.

Theorem 21 asserts that (n, s, t) -spikes of rank at least five that do not have hyperplanes of certain sizes are distinguished from all other matroids by a few numerical invariants that can be determined from the Tutte polynomial. Before treating this and related results, we first give, in Theorem 15, a necessary and sufficient condition for two (n, s, t) -spikes to have the same Tutte polynomial; in particular, we show that all (n, n, t) -spikes with the same number of circuit-hyperplanes share the same Tutte polynomial (Corollary 17). Figure 1 shows two nonisomorphic 4-spikes, each having two circuit-hyperplanes and so, according to Corollary 17, having the same Tutte polynomial.

The criterion in Theorem 15 for (n, s, t) -spikes to have the same Tutte polynomial is based on hyperplanes that do not contain the tip. It follows from condition (3) of Definition 14 that the restriction of an (n, s, t) -spike M to a hyperplane that does not contain the tip a is isomorphic to a uniform matroid $U_{n-1, h}$ for some h with $n - 1 \leq h \leq s$. For such h , let c_h^M denote the number of hyperplanes of M that do not contain the tip and for which the corresponding restrictions of M are isomorphic to $U_{n-1, h}$. In particular, the number of circuit-hyperplanes of M is given by c_n^M . Also, for the free (n, s, t) -spike M we have $c_{n-1}^M = \binom{s}{n-1}(t-1)^{n-1}$ and $c_h^M = 0$ for $n \leq h \leq s$. More generally, since any $n - 1$ points, not including the tip, chosen from distinct lines through the tip span a unique hyperplane that does not contain the tip, we have

$$\binom{s}{n-1}(t-1)^{n-1} = \sum_{h=n-1}^s c_h^M \binom{h}{n-1}.$$

Thus, any $s - n + 1$ of the numbers $c_{n-1}^M, c_n^M, \dots, c_s^M$ determine the other number in this sequence. In the following theorem we prove that any $s - n + 1$ of the numbers $c_{n-1}^M, c_n^M, \dots, c_s^M$ determine the Tutte polynomial of an (n, s, t) -spike M , and conversely.

Theorem 15. *Two (n, s, t) -spikes M and N have the same Tutte polynomial if and only if $c_h^M = c_h^N$ for any $s - n + 1$ integers h with $n - 1 \leq h \leq s$.*

Proof. By the remark above, $c_h^M = c_h^N$ for all h with $n - 1 \leq h \leq s$ if and only if $c_h^M = c_h^N$ for any $s - n + 1$ integers h with $n - 1 \leq h \leq s$. We focus on the first of these conditions.

That the numbers c_h^M determine the Tutte polynomial of M follows from the definition of the Tutte polynomial given in equation (1) once we show that we can determine the number of subsets with a given rank and cardinality solely from the numbers c_h^M and the conditions that define an (n, s, t) -spike. Let A be a subset of S . First assume that A contains the tip a or that A contains two or more points from some line ℓ_i . If A contains at least one point other than the tip from j of the lines $\ell_1, \ell_2, \dots, \ell_s$, then it follows from condition (3) of Definition 14 that the rank of A is given as follows:

$$r(A) = \begin{cases} j + 1, & \text{if } j < n; \\ n, & \text{otherwise.} \end{cases}$$

Now assume that A does not contain the tip and that A contains at most one point from each line $\ell_1, \ell_2, \dots, \ell_s$. Note that if $|A| < n$, then $r(A) = |A|$ since we have $r(A \cup a) = |A| + 1$ from condition (3) of Definition 14. All sets not yet considered have cardinality k , for some $k \geq n$, and rank $n - 1$ or n . The number of such subsets having k points is $\binom{s}{k}(t - 1)^k$, and among these, exactly

$$\sum_{h=k}^s c_h^M \binom{h}{k}$$

have rank $n - 1$.

For the converse, it follows in the same way that if two (n, s, t) -spikes M and N have the same Tutte polynomial, then for all k with $n - 1 \leq k \leq s$ we have

$$\sum_{h=k}^s c_h^M \binom{h}{k} = \sum_{h=k}^s c_h^N \binom{h}{k}. \quad (4)$$

Let a_k be the sum in Eq. (4). The matrix with rows and columns indexed by $n - 1, n, \dots, s$ whose k, h entry is $\binom{h}{k}$ is upper triangular with all 1s on the diagonal, so the system of linear equations $\sum_{h=k}^s x_h \binom{h}{k} = a_k$, with $n - 1 \leq k \leq s$, has a unique solution. Thus, from Eq. (4), we conclude that $c_h^M = c_h^N$ for all h with $n - 1 \leq h \leq s$. \square

Since the number of circuits of cardinality n is given by $\sum_{h=n}^s c_h^M \binom{h}{n}$, the following corollary is immediate.

Corollary 16. *If two (n, s, t) -spikes have the same Tutte polynomial, then they have the same number of circuits of cardinality n .*

For (n, n, t) -spikes, we have the following stronger corollary.

Corollary 17. *Two (n, n, t) -spikes have the same Tutte polynomial if and only if they have the same number of circuit-hyperplanes.*

The following extremal property of (n, n, t) -spikes will be useful.

Theorem 18. Assume $n \geq 4$. An (n, n, t) -spike has at most $(t - 1)^{n-1}$ circuit-hyperplanes. In particular, an n -spike has at most 2^{n-1} circuit-hyperplanes. The only n -spikes with 2^{n-1} circuit-hyperplanes are binary, and all binary n -spikes are isomorphic.

Proof. We already observed that each circuit-hyperplane of an (n, n, t) -spike M contains exactly one point from each set $\ell_i - a$. It follows from condition (3) of Definition 14 that for any set that contains one point from each of the sets $\ell_1 - a, \ell_2 - a, \dots, \ell_{n-1} - a$, there is at most one point from $\ell_n - a$ that completes this set to an n -circuit; this gives the claimed bound on circuit-hyperplanes. These ideas also yield the following statement.

(18.1) Assume that C is a circuit-hyperplane of an (n, n, t) -spike and ℓ_i is $\{a, x_1, x_2, \dots, x_{t-1}\}$ with $x_1 \in C$. Then for any $j \geq 2$, the set $(C - x_1) \cup x_j$ is a basis.

Assume that M is an n -spike with 3-point lines

$$\ell_1 = \{a, x_1, y_1\}, \ell_2 = \{a, x_2, y_2\}, \dots, \ell_n = \{a, x_n, y_n\}$$

and with 2^{n-1} circuit-hyperplanes. Let \mathcal{Z}_n be as defined in Eq. (3). Thus, $|\mathcal{Z}_n| = 2^n$. Since each set in \mathcal{Z}_n is either a basis or a circuit-hyperplane, it follows that there are 2^{n-1} circuit-hyperplanes in \mathcal{Z}_n and 2^{n-1} bases in \mathcal{Z}_n . By (18.1), for any i with $1 \leq i \leq n$, the map that takes X to the symmetric difference $X \Delta \{x_i, y_i\}$ is a bijection of \mathcal{Z}_n that maps circuit-hyperplanes to bases. Therefore we get the following statement.

(18.2) If B is a basis of M in \mathcal{Z}_n , then the symmetric difference $B \Delta \{x_i, y_i\}$ is a circuit-hyperplane.

From (18.1) and (18.2), we get the following statement.

(18.3) For a circuit-hyperplane C of M and any integers $1 \leq i_1 < i_2 < \dots < i_k \leq n$, we have that the symmetric difference

$$C \Delta \{x_{i_1}, y_{i_1}\} \Delta \{x_{i_2}, y_{i_2}\} \Delta \dots \Delta \{x_{i_k}, y_{i_k}\}$$

is a circuit-hyperplane if and only if k is even; otherwise this symmetric difference is a basis.

Using (18.3), one can easily construct an isomorphism between any two n -spikes with 2^{n-1} circuit-hyperplanes. Using this and the fact that the n -spike represented by matrix D in Eq. (2) has 2^{n-1} circuit-hyperplanes, it follows that any n -spike with 2^{n-1} circuit-hyperplanes is binary. Alternatively, using the Scum Theorem [20, Proposition 3.3.7] and counting, it is easy to check that any n -spike with 2^{n-1} circuit-hyperplanes has no $U_{2,4}$ -minor and so is binary. Similarly, it follows that any binary n -spike has 2^{n-1} circuit-hyperplanes. \square

Although it is not directly relevant to the other results in this paper, in Theorem 20 we present an extension of the results in Theorem 18 on binary n -spikes to $(n, n, q + 1)$ -spikes that are representable over the finite field $\text{GF}(q)$. In particular, Theorem 20 shows that if $t - 1$ is a prime power, then the bound of $(t - 1)^{n-1}$ in Theorem 18 is tight. The proof of Theorem 20 rests on Lemma 19, which is a matroid-theoretic reformulation of what is often called the fundamental theorem of projective geometry [15, Section 2.1.2]. (See [5, Lemma 9] for a matroid-theoretic proof of this lemma.)

Lemma 19. *Let $B = \{b_1, b_2, \dots, b_n\}$ be a basis of $\text{PG}(n - 1, q)$ and let b be a point in $\text{PG}(n - 1, q)$ such that the fundamental circuit $C(b, B)$ of b with respect to the basis B is $B \cup b$. Let $B' = \{b'_1, b'_2, \dots, b'_n\}$ be a basis of $\text{PG}(n - 1, q)$ and let b' be a point in $\text{PG}(n - 1, q)$ such that $C(b', B') = B' \cup b'$. Then there is an automorphism ϕ of $\text{PG}(n - 1, q)$ such that $\phi(b_i) = b'_i$ for $i = 1, 2, \dots, n$ and $\phi(b) = b'$.*

Theorem 20. *Assume $n \geq 4$. Let q be a prime power. Up to isomorphism, there is a unique $(n, n, q + 1)$ -spike that is representable over $\text{GF}(q)$. This $(n, n, q + 1)$ -spike has q^{n-1} circuit-hyperplanes.*

Proof. We first construct an $(n, n, q + 1)$ -spike that is representable over $\text{GF}(q)$ and that has q^{n-1} circuit-hyperplanes. Let $B = \{b_1, b_2, \dots, b_n\}$ be a basis of the rank- n projective geometry $\text{PG}(n - 1, q)$. Let a be a point of $\text{PG}(n - 1, q)$ so that the fundamental circuit $C(a, B)$ of a with respect to B is $B \cup a$. Let ℓ_i be the closure $\text{cl}_P(\{a, b_i\})$ of $\{a, b_i\}$ in $\text{PG}(n - 1, q)$, and let M be the restriction $\text{PG}(n - 1, q)|(\ell_1 \cup \ell_2 \cup \dots \cup \ell_n)$. Clearly M is an $(n, n, q + 1)$ -spike. To see that M has q^{n-1} circuit-hyperplanes, note that there are q^{n-2} sets $X = \{x_1, x_2, \dots, x_{n-2}\}$ for which x_i is in $\ell_i - a$ for $1 \leq i \leq n - 2$. Note that each such set X is a flat of M of rank $n - 2$. Note also that each such set X is contained in at least $q + 1$ distinct hyperplanes of M , namely $\text{cl}_M(X \cup \{y\})$ for each of the $q + 1$ points y of ℓ_{n-1} . Since a rank- $(n - 2)$ flat of a restriction of $\text{PG}(n - 1, q)$ is contained in at most $q + 1$ distinct hyperplanes, there are no other hyperplanes of M that contain X . It follows that for each point z of $\ell_n - a$, there is a point y of $\ell_{n-1} - a$ such that $z \in \text{cl}_M(X \cup \{y\})$. Thus, each of the q^{n-2} sets X is contained in q circuit-hyperplanes of M , so M has q^{n-1} circuit-hyperplanes.

To prove the uniqueness assertion, note that if M is an $(n, n, q + 1)$ -spike that is a restriction of $\text{PG}(n - 1, q)$, then for the apex a of M and for any basis $B = \{b_1, b_2, \dots, b_n\}$ of M with $b_i \in \ell_i - a$ for i with $1 \leq i \leq n$, we have $C(a, B) = B \cup a$ by condition (3) of Definition 14. For a second such $(n, n, q + 1)$ -spike N and any choice of such a basis of N , the automorphism ϕ of Lemma 19 maps M to N . Thus, M and N are isomorphic. \square

Theorem 20 generalizes the result (Theorem 18) that there is a unique binary $(n, n, 3)$ -spike by showing that there is a unique $(n, n, q + 1)$ -spike that is representable over $\text{GF}(q)$. The situation is very different for large n and q if instead we focus on $t = 3$, rather than $t = q + 1$; Wu [25] shows that the number of nonisomorphic $(n, n, 3)$ -spikes that are representable over $\text{GF}(q)$ grows rapidly as n and q increase.

We now show that certain classes of spikes can be detected by a few numerical invariants that can be determined from the Tutte polynomial; condition (6) of Theorem 21 is what mildly limits the scope of this result.

Theorem 21. Assume n , s , and t are integers with $n \geq 5$, $s \geq n - 1$, and $t \geq 3$. Assume that M is a rank- n geometry that has:

- (1) $s(t - 1) + 1$ points,
- (2) s lines that each have exactly t points,
- (3) $s \binom{t}{3}$ circuits with three elements,
- (4) $\binom{s}{2} \binom{t-1}{2}^2$ circuits with four elements,
- (5) $\binom{s}{2}$ planes with $2t - 1$ points, and
- (6) for each j with $j \geq n - 1$, no hyperplane with $j(t - 1) + 1$ points.

Then M is an (n, s, t) -spike. If in addition there are no n -circuits, then M is the free (n, s, t) -spike. If $s = n$, $t = 3$, and M has 2^{n-1} circuits with n elements, then M is the binary n -spike.

Proof. Assumptions (2) and (3) imply that the t -point lines are the only nontrivial lines. Note that $\binom{t-1}{2}^2$ is the minimum number of 4-circuits in a plane that has $2t - 1$ points in which each line has either 2 or t points; furthermore, the only such plane that has $\binom{t-1}{2}^2$ circuits with four elements is the parallel connection of two t -point lines. Since M has $\binom{s}{2}$ planes with $2t - 1$ points and $\binom{s}{2} \binom{t-1}{2}^2$ circuits with four elements, it follows that each $(2t - 1)$ -point plane of M is a parallel connection of two t -point lines. Furthermore, since there are $\binom{s}{2}$ planes of M that have $2t - 1$ points, each of the $\binom{s}{2}$ pairs of t -point lines spans one of these planes and therefore has nonempty intersection. Since $n \geq 5$, this implies that all t -point lines contain some common point, say a . Thus, conditions (1) and (2) in Definition 14 hold.

From assumption (6) it follows that each hyperplane of M that contains a can contain at most $n - 2$ of the t -point lines. This implies that, for $i \leq n - 1$, any rank- i flat of M that contains a can contain at most $i - 1$ of the t -point lines. Thus, condition (3) in Definition 14 holds, so M is an (n, s, t) -spike.

The assertion about free (n, s, t) -spikes follows immediately; that about binary n -spikes follows from Theorem 18. \square

Let $\mathcal{S}_{n,s,t}^k$ be the set of all (n, s, t) -spikes that satisfy condition (6) in Theorem 21 for which the number of n -circuits is exactly k . Matroids in $\mathcal{S}_{n,s,t}^k$ are distinguished from all other matroids by their Tutte polynomials, as the following corollary states.

Corollary 22. Assume n , s , and t are integers with $n \geq 5$, $s \geq n - 1$, and $t \geq 3$. If N is an (n, s, t) -spike in $\mathcal{S}_{n,s,t}^k$ and $t(M; x, y) = t(N; x, y)$, then M is an (n, s, t) -spike in $\mathcal{S}_{n,s,t}^k$. In particular, if N is the only (n, s, t) -spike in $\mathcal{S}_{n,s,t}^k$, then N is T -unique.

Proof. From Theorems 2 and 3, it follows that M satisfies conditions (1)–(6) in Theorem 21. Thus, M is an (n, s, t) -spike. That k is the number of n -circuits in M follows from Theorem 9 and the fact that N is $(n - 1)$ -chordal, or, alternatively, from Corollary 16. \square

Note that condition (6) in Theorem 21 is automatically satisfied by any (n, s, t) -spike with $s < (n - 1)(t - 1) + 1$. In particular, the first assertion in Corollary 22 applies to all (n, n, t) -spikes. From Corollary 22, we also get the T-uniqueness of some families of (n, s, t) -spikes.

Corollary 23. *For integers n , s , and t with $n \geq 5$, $s \geq n - 1$, and $t \geq 3$, the following matroids are T-unique:*

- (1) *the free (n, s, t) -spike,*
- (2) *the binary n -spike, and*
- (3) *the (n, s, t) -spike M where, for some integer h with $n \leq h \leq s$ and h not of the form $j(t - 1) + 1$ for $j \geq n - 1$, we have*

$$c_k^M = \begin{cases} 1, & \text{if } k = h; \\ 0, & \text{if } n \leq k \leq s \text{ and } k \neq h. \end{cases}$$

That the hyperplanes of a $(4, s, t)$ -spike isomorphic to $U_{3,h}$, for $h \geq 4$, contain 4-circuits makes the argument about the structure of $(2t - 1)$ -point planes in the proof of Theorem 21 fail in general for $n = 4$. However, the same ideas as appear in the proofs of Theorem 21 and Corollary 22 give the following result.

Theorem 24. *The free $(4, s, t)$ -spike is T-unique.*

6. An application to matroid reconstruction

Graph reconstruction problems have interesting matroid counterparts [9,19,20]. We focus on reconstruction from hyperplanes and from single-element deletions. The *deck of hyperplanes* of a matroid M is the multiset of unlabeled hyperplanes. That is, for each isomorphism type H of rank $r(M) - 1$, we know the number of hyperplanes of M that are isomorphic to H . A matroid M is *hyperplane reconstructible* if only matroids that are isomorphic to M have the same deck of hyperplanes as M . Similarly, the *deck of single-element deletions* of a matroid M is the multiset of unlabeled single-element deletions. A matroid M is *deletion reconstructible* if only matroids that are isomorphic to M have the same deck of single-element deletions as M . Hyperplane reconstructible matroids are also deletion reconstructible (see [19]). Projective and affine geometries of rank four or more are known to be hyperplane reconstructible, as are the cycle matroids of complete graphs, and, more generally, Dowling lattices of rank four or more (see [4]). In [1], the geometries $\text{PG}(n - 1, q) \setminus \text{PG}(k - 1, q)$, for $n > 3$ and $1 \leq k \leq n - 2$, are shown to be hyperplane reconstructible. Brylawski [9] showed that the Tutte polynomial of a matroid can be

computed from the deck of hyperplanes. From Brylawski's theorem and results in [14], it follows that truncations of projective and affine geometries are hyperplane reconstructible. It follows from results in [18] that the cycle matroids of complete bipartite graphs and the truncations of the cycle matroids of complete graphs are hyperplane reconstructible. From Brylawski's result and Theorems 13 and 24, and Corollary 23, we get the following corollary.

Corollary 25. *The following matroids are hyperplane reconstructible and deletion reconstructible:*

- (1) the (n, t) -wheel $\mathcal{W}_{n,t}$ with $n, t \geq 3$,
- (2) the (n, t) -whirl $\mathcal{W}^{n,t}$ with $n, t \geq 3$,
- (3) the free (n, s, t) -spike with $n \geq 4$, $s \geq n - 1$, and $t \geq 3$,
- (4) the binary n -spike with $n \geq 5$, and
- (5) the (n, s, t) -spike M where $n \geq 5$, $s \geq n - 1$, $t \geq 3$, and for some integer h with $n \leq h \leq s$ and h not of the form $j(t - 1) + 1$ for $j \geq n - 1$, we have

$$c_k^M = \begin{cases} 1, & \text{if } k = h; \\ 0, & \text{if } n \leq k \leq s \text{ and } k \neq h. \end{cases}$$

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